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Nonlinear dynamics of extensible fluid-conveying pipes, supported at both ends ☆

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Abstract

In this paper, the post-divergence behaviour of extensible fluid-conveying pipes supported at both ends is studied using the weakly nonlinear equations of motion of Semler, Li and Païdoussis. The two coupled nonlinear partial differential equations are discretized via Galerkin's method and the resulting set of ordinary differential equations is solved either by Houbolt's finite difference method or via AUTO. Typically, the pipe is stable at its original static equilibrium position up to the flow velocity where it loses stability by static divergence via a supercritical pitchfork bifurcation. The amplitude of the resultant buckling increases with increasing flow, but no secondary instability occurs beyond the pitchfork bifurcation. The effects of the system parameters on pipe behaviour as well as the possibility of a subcritical pitchfork bifurcation have also been studied.
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Keywords: Pipe conveying fluid; Supported ends; Divergence; Post-divergence flutter; Stability; Secondary bifurcations

1. Introduction

The question of existence of post-divergence flutter of pipes conveying fluid is posed exclusively for structures with supported ends; these are "inherently conservative systems", i.e. systems which, if dissipative forces are ignored, are *gyroscopic conservative*, with the gyroscopic forces doing no work (Païdoussis, 1998). It has been known for a long time that, for sufficiently high flow, such pipes lose stability by divergence. This has been confirmed by experiments by Dodds and Runyan (1965) and others. Then, in the 1970s, Païdoussis and Issid (1974) found that linear theory predicts the existence of post-divergence coupled-mode flutter, at higher flow velocities. On the other hand, the work done by the fluid on the pipe in a presumed cycle of oscillation is zero.[2] This, then, constitutes a paradox: for how is it possible for

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[2]This is strictly true for *periodic* oscillations, whilst coupled-mode flutter at its very onset is not quite periodic, as recently pointed by Kuiper (2008).

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the pipe to flutter if the system is conservative and no energy is supplied to sustain the oscillation (Done and Simpson, 1977; Païdoussis, 1998, 2005)? To resolve this paradox, the system must be analysed by a nonlinear model to ascertain the reliability of the predicted post-divergence behaviour, as linear theory is valid only up to the onset of the first instability, in this case divergence.

Nonlinear aspects of the behaviour of pipes with supported ends were first studied by Thurman and Mote (1969); they consider an axially extensible pipe and therefore the nonlinearities in their equations are associated with the axial elongation and the resultant tension in the pipe. Holmes (1978) studied this problem by simply adding one, essential, nonlinear term due to this extension-induced tension to the linear equation; he used this equation to show that pipes supported at both ends cannot flutter.

Yoshizawa et al. (1985, 1986) have studied the dynamics of an inextensible clamped–hinged pipe, which could slide axially at its hinged downstream end. They have shown that the pipe buckles at a certain flow velocity and the amplitude of buckling increases thereafter with no sign of any dynamic instability. However, as found by the present authors, the maximum flow velocity studied by Yoshizawa et al., i.e. $U = 7.2$ (where U is defined in Eq. (3)), is less than the critical flow for the onset of coupled-mode flutter predicted by the linear theory ($U = 9.0$), indicating that their calculations do not prove the non-existence of dynamic instabilities in the system—which was not the aim of their paper, in any case.

Semler et al. (1994) have derived a set of nonlinear equations of motion correct to third order of magnitude for an extensible fluid-conveying pipe supported at both ends. The corrected version of these equations may be found in Païdoussis (2004, Appendix T). To-date, this is the most complete set of weakly nonlinear equations of motion available for this system. Recently, Nikolić and Rajković (2006) used Lyapunov–Schmidt reduction and singularity theory to study the behaviour of this system in the vicinity of the bifurcation points by using the Thurman and Mote, Holmes, and Semler et al./Païdoussis models. For certain ranges of parameters, they found that a subcritical pitchfork bifurcation is possible in the two models other than Holmes' simplified model. However, Nikolić and Rajković's results are confined to the neighbourhood of the bifurcation points, and hence the question of possible post-divergence flutter is not really addressed.

In the present paper, we use the complete nonlinear model of Païdoussis (2004), to investigate the possibility of post-divergence instabilities in pipes supported at both ends.

2. Equations of motion and the method of solution

The system under consideration, as shown in Fig. 1, consists of a pipe with supported ends (clamped ends in the figure) conveying fluid. The pipe, of length L, mass per unit length m and flexural rigidity EI, is generally vertically mounted, although idealized horizontal systems are also considered in which gravity and gravity-induced sag are neglected. The fluid, of mass M per unit length flows generally downwards with a mean steady velocity U^*. The system may be subject to an externally applied tension $\bar{T}$ (say, by moving the two supports apart) and a mean pressurization *vis-à-vis* the external ambient fluid, $\bar{P}$; this pressure is over and above the pressure needed to overcome frictional pressure drag, which is a maximum at the upstream end and zero at the downstream one.

In the derivation of equations of motion [see Païdoussis (2004, Appendix T.4)] the flow is presumed to be a fully developed turbulent flow; it is found, however, that for the simple plug-flow model used, the viscous pressure-loss terms exactly cancel the friction-related traction (tension) terms acting on the pipe, and none of these viscosity/friction-related terms appear in the final equations of motion.

In the derivation of the equations of motion, the pipe is considered to be extensible. Furthermore, it has been assumed that transverse displacements, $v^*(x,t)$ are sufficiently small, of order $\mathcal{O}(\varepsilon)$, for the axial displacements, $u^*(x,t)$ to be of second order, $\mathcal{O}(\varepsilon^2)$. Moreover, motions are assumed to be planar, in the (x,y)-plane. The resultant equations of motion (Païdoussis, 2004, Appendix T.4), are

$$\ddot{u} + 2U\sqrt{\beta}\dot{u}' + U^2 u'' - \Pi_0 u'' - (v''v''' + v'v^{(4)}) - \gamma\left[\tfrac{1}{2}v'^2 - (1-\xi)v'v''\right] + (\Gamma - \Pi_0 - \Pi)v'v'' + \mathcal{O}(\varepsilon^4) = 0, \quad (1)$$

$$\begin{aligned} &\ddot{v} + 2U\sqrt{\beta}\dot{v}' + U^2 v'' - (\Gamma - \Pi)v'' + v'''' + \gamma v' + (\Gamma - \Pi_0 - \Pi)(u''v' + u'v'' + \frac{3}{2}v'^2 v'') \\ &- (8v'v''v''' + v'u^{(4)} + 2v'^2 v^{(4)} + 2v''^3 + 2u'v^{(4)} + 4u''v''' + 3u'''v'') \\ &-\gamma\left[u'v' + \frac{1}{2}v'^3 - (1-\xi)(-v'' + v'u'' + v''u' + \frac{3}{2}v'^2 v'')\right] + \mathcal{O}(\varepsilon^4) = 0, \end{aligned} \quad (2)$$

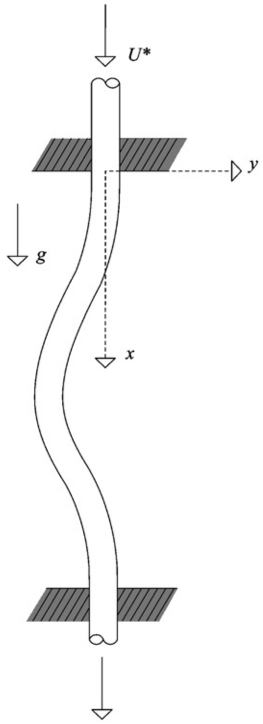

Fig. 1. A vertical pipe conveying fluid with supported ends.

thus neglecting terms of $\mathcal{O}(\varepsilon^4)$ and higher. In these equations, u and v are dimensionless displacements of the pipe in the axial and the transverse direction, respectively, $u = u^*/L$, $v = v^*/L$; U is the dimensionless flow velocity, γ the dimensionless gravity parameter, β a dimensionless mass parameter, Π_0 the dimensionless axial flexibility, Γ the dimensionless tension, and Π the dimensionless pressurization. These dimensionless parameters are related to the dimensional ones by the following relations:

$$U = \left(\frac{M}{EI}\right)^{1/2} U^* L, \quad \beta = \frac{M}{m+M}, \quad \gamma = \frac{m+M}{EI} L^3 g, \quad \Pi_0 = \frac{EAL^2}{EI}, \quad \Gamma = \frac{\bar{T}L^2}{EI}, \quad \Pi = \frac{\bar{P}AL^2}{EI}. \tag{3}$$

The equations of motion are discretized via Galerkin's method, using as basis functions the eigenfunctions of a beam for the transverse deformation and those of a bar for the axial deformation. The resulting ordinary differential equations are

$$M^u_{ij}\ddot{p}_j + C^u_{ij}\dot{p}_j + K^u_{ij}p_j + A_{ijk}q_j q_k = 0, \tag{4}$$

$$M^v_{ij}\ddot{q}_j + C^v_{ij}\dot{q}_j + K^v_{ij}q_j + D_{ijk}p_j q_k + F_{ijkl}q_j q_k q_l = 0, \tag{5}$$

in which M^u_{ij}, C^u_{ij} and K^u_{ij} are the linear mass, damping and stiffness coefficients in the axial direction, and M^v_{ij}, C^v_{ij} and K^v_{ij} are the corresponding linear terms in the transverse direction; A_{ijk}, D_{ijk} and F_{ijkl} are nonlinear coefficients (see Appendix A). The size of this set of ordinary differential equations depends on the number of Galerkin modes used in the discretization.

To solve this system of equations, we have used either Houbolt's finite difference method (Semler et al., 1996) or AUTO, which is a software package for continuation and bifurcation analysis of ordinary differential equations (Doedel, 2007). To be able to use AUTO, one has to write the set of the second-order equations, Eqs. (4) and (5), in first-order form:

$$\dot{Y} = [A]Y, \tag{6}$$

in which $[A]$ is an $N_t \times N_t$ matrix, where $N_t = N_u + N_v$, and N_u and N_v are the numbers of Galerkin modes used in the axial and the transverse direction, respectively. The elements of $[A]$ are constant linear terms and terms depending on the unknowns arising from the nonlinear terms of the partial differential equations; Y is the vector containing the dimensionless generalized coordinates and their first time-derivatives.

The results to be presented have been obtained with 6 basis modes in each direction, i.e. $N_u = N_v = 6$. Convergence tests were conducted, up to the highest flow velocities used (convergence is generally a function of U), and it was found that the critical flow velocities converge to their final values when 3 or more modes are used in each direction and the bifurcation diagrams lie on top of each other when 6 or more modes in each direction are used.

3. Post-divergence behaviour

Fig. 2(a) shows the bifurcation diagram for a simply supported pipe with the parameters listed in Table 1. The ordinate in this bifurcation diagram is the dimensionless midpoint transverse displacement of the pipe, while the abscissa is the dimensionless flow velocity. Continuous lines represent stable static solutions and dotted lines unstable ones. The pipe is stable and at its original static equilibrium position, up to $U = \pi$ (the first circle in the bifurcation diagram of Fig. 2(a)), where it loses stability by a supercritical pitchfork bifurcation. It is seen that for $U > \pi$ there are three possible solutions: an unstable one, corresponding to the original static equilibrium (dotted line along the abscissa), and two stable attracting solutions, corresponding to buckling of the pipe to either side of the vertical. The pipe can be attracted to one or the other of these latter solutions, depending on the initial conditions utilized. Increasing the flow velocity beyond $U = \pi$, the amplitude of buckling increases, but no secondary bifurcation (static or dynamic)

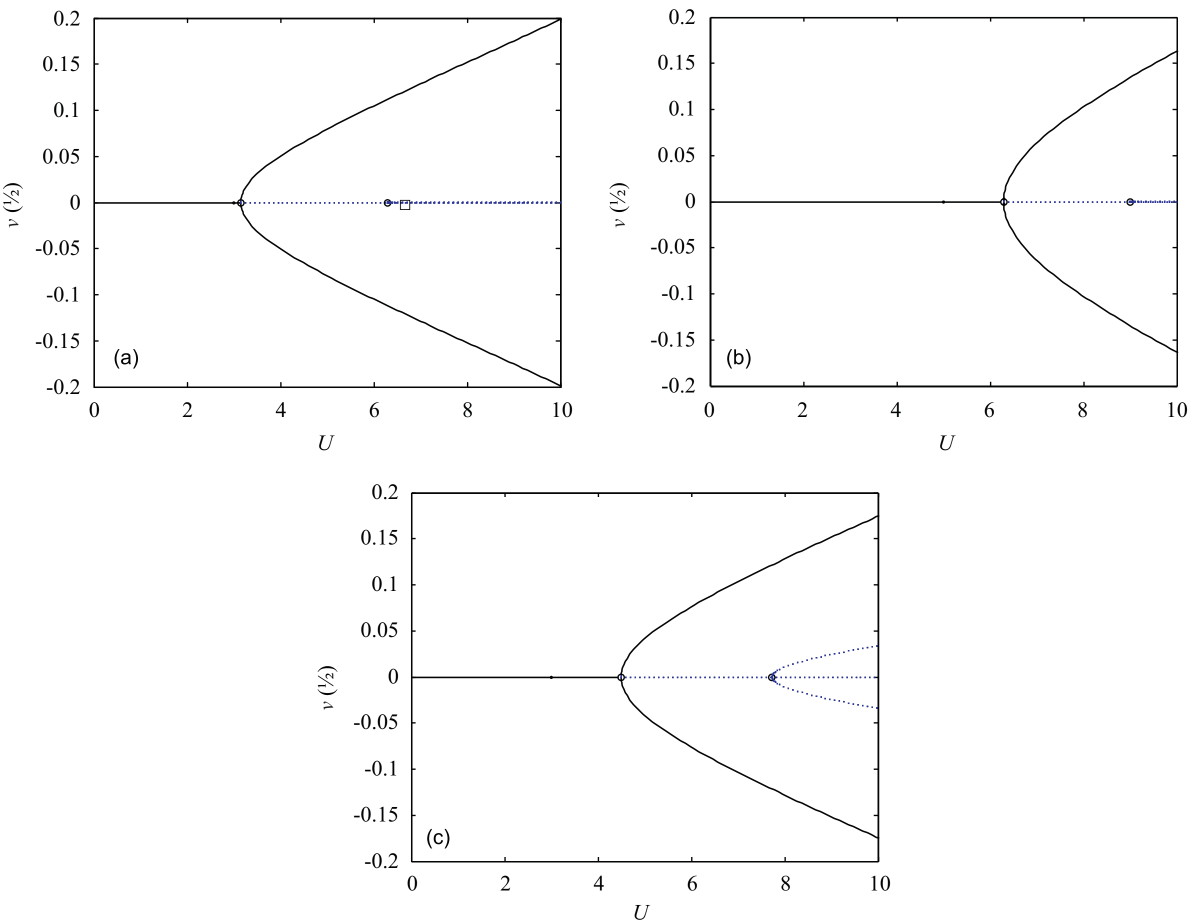

Fig. 2. Bifurcation diagram for the (a) hinged-hinged supported, (b) clamped–clamped, and (c) clamped–hinged pipe conveying fluid with parameters as in Table 1, where $v(1/2)$ is the dimensionless transverse displacement at the middle of the pipe.

Table 1
Dimensionless parameters of the pipe.

Nondimensional parameter	γ	β	Π_0	Γ	Π
Numerical value	0.1	0.47	1000	0	0

materializes. This is what Holmes (1978) concluded by using his simplified equation of motion for pipes supported at both ends. The bifurcation diagram of Fig. 2(a) shows another pitchfork bifurcation at $U = 2\pi$, in the range of flow shown; this static instability, together with the coupled-mode flutter, which was predicted by the linear theory of Païdoussis and Issid (1974) (□ in Fig. 2(a)), happen where the solution is already unstable (dotted lines in the bifurcation diagram) and therefore do not materialize.

Bifurcation diagrams of a pipe with clamped–clamped or clamped–hinged boundary conditions are given in Fig. 2(b) and (c), respectively, with a behaviour qualitatively similar to that of the simply supported pipe and with no sign of dynamic instability.

Here it should be remarked that, unlike the situation with internal flow, post-divergence flutter does arise for *external axial flow*—over the pipe (cylinder), as seen in experiments and proved by nonlinear theory (Modarres-Sadeghi et al., 2005, 2008). This difference in dynamical behaviour has been found to be related to the frictional terms. The dynamics of the pipe problem (internal flow) has been shown (Benjamin, 1961; Gregory and Païdoussis, 1966; Païdoussis, 1998) to be independent of fluid frictional forces (see Eqs. (1) and (2)), since they are exactly counterbalanced by the pressure-loss forces along the pipe; as a result, both pressure-loss and frictional forces vanish from the equation of motion. This is not true for external flow over a cylinder, in which the mean pressure is largely unaffected by frictional effects on the cylinder surface. Some of these frictional terms give rise to terms in the equation of motion of a different derivative form *vis-à-vis* the inviscid-force terms, and hence *vis-à-vis* the internal flow case.

4. The influence of different parameters on the stability and the amplitude of the buckled solution

There are five system parameters in the equations of motion, Eqs. (1) and (2): γ, β, Π_0, Γ, and Π, other than U which is utilized as the bifurcation parameter. In this section, their influence on the behaviour of a hinged–hinged pipe is studied. Similar behaviour is expected for clamped–clamped and clamped–hinged boundary conditions.

4.1. *Influence of the dimensionless axial flexibility, Π_0*

The dimensionless axial flexibility, $\Pi_0 = EAL^2/EI$, is a measure of the axial rigidity as compared with the transverse rigidity of the system. For a pipe of length L, inner diameter of D_i and outer diameter of D_o, the dimensionless axial rigidity is $\Pi_0 = 16L^2/(D_o{}^2 + D_i^2)$. Thus, for fixed inner and outer diameters, increasing Π_0 means a longer and hence more flexible pipe. Therefore, one would expect divergence to occur earlier and with a larger buckling amplitude.

Fig. 3(a) shows the bifurcation diagrams of the system for three different values of Π_0, for $\beta = 0.47$ and $\gamma = \Pi = \Gamma = 0$, where q_1 is the first generalized coordinate in the transverse direction and is representative of the cylinder displacement in this direction. It is seen that the larger the value of Π_0 is, the smaller the amplitude of buckling becomes, which seems to be contrary to what was expected! The answer to this paradox lies in the fact that the dimensionless flow velocity, U, and the dimensionless transverse displacement, v, both depend on the length of the pipe. Once, this is taken into account, the results are as one would expect: a longer pipe buckles at lower *dimensional* flow speed; and when buckled, for a given flow velocity, its amplitude is larger than for a shorter pipe.

The dimensionless critical flow velocity for the pitchfork bifurcation does not change with Π_0. This is the case only when $\gamma = 0$. If gravity is taken into account, as γ is a function of the pipe length, by increasing Π_0 the value of γ increases and, as a result, the critical flow velocity for the pitchfork bifurcation becomes larger (see Section 4.3).

4.2. *Influence of the dimensionless mass parameter, β*

As may be verified in Eqs. (1) and (2), the parameter β is always associated with pipe-velocity-related terms. Hence, as the system loses stability by divergence and the post-buckling behaviour remains static, the parameter β does not play a role on the critical point or post-critical behaviour of the system.

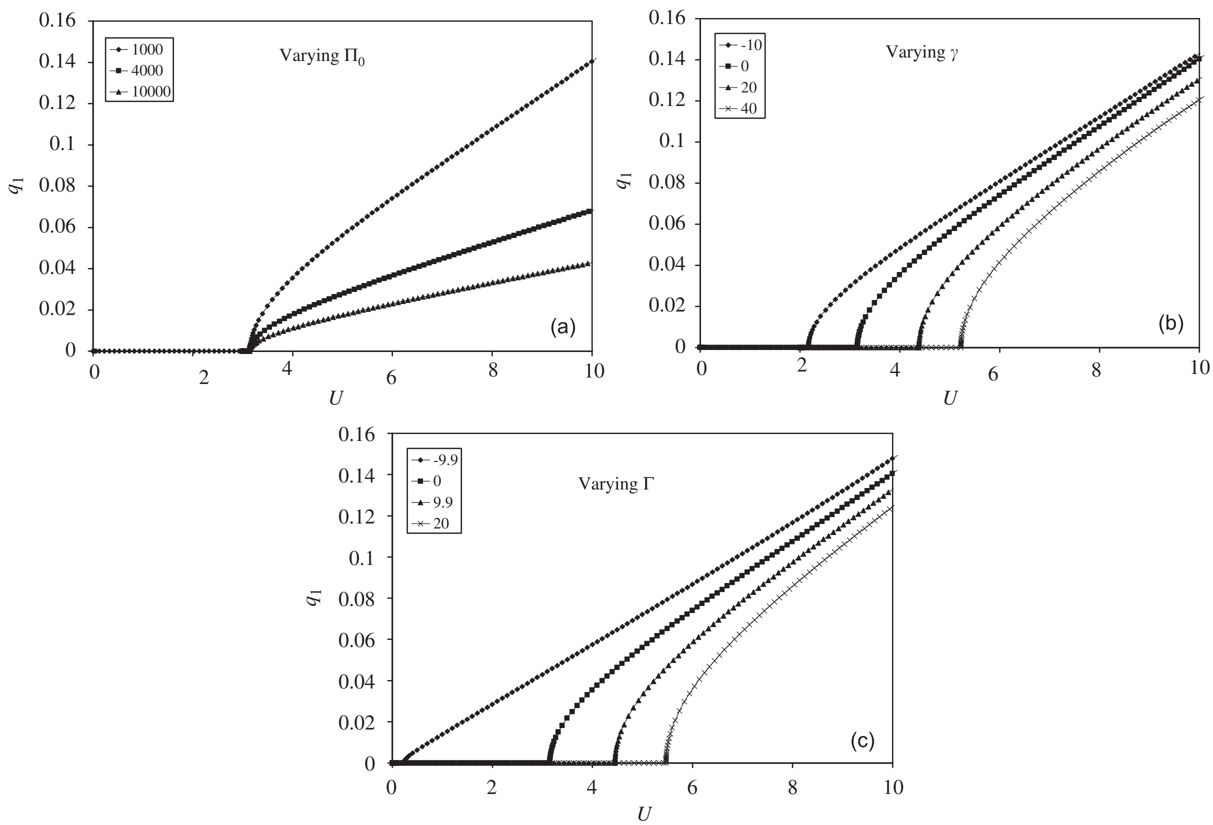

Fig. 3. Bifurcation diagrams for the system with different values of (a) the nondimensional axial rigidity parameter, Π_0, with no gravity, (b) nondimensional gravity parameter, γ, and (c) tension parameter, Γ. The other parameters are as in Table 1.

4.3. Influence of the dimensionless gravity parameter, γ

An idealized horizontal system (no sag) is characterized by $\gamma = 0$; while for vertical systems, $\gamma \neq 0$. A positive value of γ means that the flow vector is in the same direction as gravity (Fig. 1), that is, downward; in this case the pipe is subjected to gravity-induced tension. When $\gamma < 0$, the gravity and flow vectors are in opposite directions (this is the case of a so-called "up-standing pipe"); the pipe in this case is under gravity-induced compression. This helps in the interpretation of the results presented in Fig. 3(b). The critical dimensionless flow velocity increases by increasing γ. The critical value of γ required for the system to buckle under its own weight is, $\gamma = -18$, in agreement with the linear results of Païdoussis and Issid (1974). Moreover, by increasing the gravity parameter while keeping the flow velocity fixed, the amplitude of the buckled pipe decreases.

4.4. Influence of the dimensionless tension/compression, Γ

When the pipe conveying fluid is under an external tension, the critical flow velocity for divergence is higher than that for no external tension. Fig. 3(c) shows the bifurcation diagrams for the pipe with different values of externally imposed tension. By increasing the external tension, the critical flow velocity for divergence increases; also, at a fixed flow velocity, the amplitude of buckling is decreased. A pipe under compression behaves in the opposite way: a smaller critical flow velocity and larger amplitude at a fixed flow velocity are obtained, as the compression is increased.

Nikolić and Rajković (2006) obtained a closed form relation for all the critical flow velocities of the pitchfork bifurcations. The critical value for the first bifurcation point, according to their calculations, is

$$U_{\rm cr} = \pi + \frac{\gamma + 2(\Gamma - \Pi)}{4\pi} + \mathcal{O}\left(\left(\frac{\Gamma - \Pi}{2\pi}\right)^2\right). \tag{7}$$

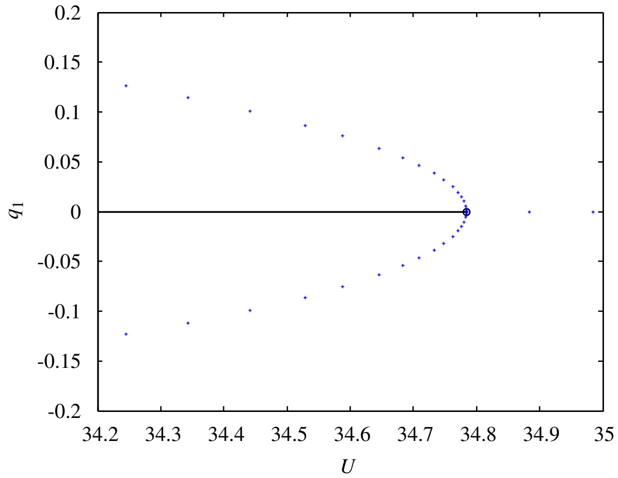

Fig. 4. Subcritical pitchfork bifurcation for the pipe under tension, $\Gamma = 1200$.

The critical flow velocities for divergence found in this paper agree with what can be found by using Eq. (7), for small values of $\Gamma - \Pi$.[3]

5. On the possibility of a subcritical pitchfork bifurcation

Nikolić and Rajković (2006) showed that, for Thurman and Mote's (1969) as well as Païdoussis' (2004) model, the pitchfork bifurcation of the system is subcritical if

$$\Gamma - \Pi > \Pi_0. \tag{9}$$

They observed that there is no possibility for a subcritical pitchfork bifurcation in Holmes' model and concluded that the nonlinear terms in the axial-direction equations in the other models make the subcritical bifurcation possible, if the relation above is satisfied.

Here, we use the full set of Eqs. (1) and (2) to investigate the possible existence of a subcritical pitchfork bifurcation in this system, numerically. Fig. 4 shows the bifurcation diagram of the system for $\Pi_0 = 1000$, $\Pi = 0$ and $\Gamma = 1200$. These parameters satisfy the necessary condition for the subcritical pitchfork bifurcation derived by Nikolić and Rajković, Eq. (9). The system is stable up to $U = 34.78$, at which point the pipe loses its stability and two unstable branches arise thereafter. These two unstable branches, which go backward, are a sign of a subcritical pitchfork bifurcation. One would expect these branches to reach a limit point at some flow velocity $U < 34.78$ and then fold, becoming stable and moving to the right; this then would give rise to the possibility of a sudden jump from static equilibrium to a relatively large amplitude of buckling, in a range of flow velocity around $U = 34$. This, however, was not obtained in the numerical results. In any case, as the unstable solution goes backward, its amplitude increases up to a point where it becomes larger than unity, indicating a displacement at the midpoint of the pipe greater than the pipe length: physically impossible! Thus, although it has been shown that a subcritical bifurcation is indeed possible, a stable solution at higher flow velocity could not be obtained, most probably because the weakly nonlinear analytical model loses its validity before a stable solution is reached.

7. Conclusions

The post-divergence dynamical behaviour of a pipe with supported ends conveying fluid is studied by using the nonlinear model based on the equations of motion of Semler, Li and Païdoussis (Païdoussis, 2004), accounting for the pipe extensibility. In this model, two coupled nonlinear partial differential equations describe the motion of the pipe. Pipes with three different boundary conditions were studied: hinged–hinged, clamped–clamped and clamped–hinged.

[3]Of course Eq. (7) can be used only when $\Gamma - \Pi$ is negligible. Therefore, for the cases of large $\Gamma - \Pi$, e.g., the results shown in Fig. 3(c) with $\Pi = 0$ and $\Gamma = -9.9$, 9.9 and 20, Eq. (7) is not applicable.

Typically, the pipe remains in its undeformed static equilibrium state at low flow velocities and then undergoes a static pitchfork bifurcation at a critical flow velocity, which depends on the system parameters. In all cases, the amplitude of the buckled pipe increases with flow, but no secondary bifurcation is observed, in agreement with Holmes' (1978) remarkable finding.

The influence of the system parameters on the critical point for divergence and the amplitude of buckling, as well as the possible existence of subcritical pitchfork bifurcations were also studied.

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Appendix A. The coefficients in Eqs. (4) and (5)

The coefficients of the tensor form of the equations of motion, Eqs. (4) and (5), are given here:

$$M^u_{ij} = \int_0^1 \psi_i\psi_j \, d\xi, \quad C^u_{ij} = 2U\sqrt{\beta}\int_0^1 \psi_i\psi'_j \, d\xi, \quad K^u_{ij} = U^2\int_0^1 \psi_i\psi''_j \, d\xi - \Pi_0\int_0^1 \psi_i\psi'' \, d\xi,$$

$$A_{ijk} = \int_0^1 \left[-(\psi_i\phi''_j\phi'''_k + \psi_i\phi'_j\phi^{(4)}_k) - \tfrac{1}{2}\gamma\psi_i\phi'_j\phi' + \gamma(1-\xi)\psi_i\phi'_j\phi''_j + (\varGamma - \varPi_0 - \varPi)\psi_i\phi'_j\phi''\right] d\xi,$$

$$M^v_{ij} = \int_0^1 \phi_i\phi_j \, d\xi, \quad C^v_{ij} = 2U\sqrt{\beta}\int_0^1 \phi_i\phi'_j \, d\xi,$$

$$K^v_{ij} = \int_0^1 \left[U^2\phi_i\phi''_j - (\varGamma - \pi)\phi_i\phi''_j + \phi_i\phi^{(4)}_j + \gamma\phi_i\phi'_j - \gamma(1-\xi)\phi_i\phi''\right] d\xi,$$

$$D_{ijk} = \int_0^1 \Big\{(\varGamma - \varPi_0 - \varPi)(\phi_i\psi''_j\phi'_k + \phi_i\psi'_j\phi''_k) - (\phi_i\phi'_k\psi^{(4)}_j + 2\phi_i\psi'_j\phi^{(4)}_k + 4\phi_i\psi''_j\phi'''_k + 3\phi_i\psi'''_j\phi''_k)$$
$$-\gamma[\phi_i\psi'_j\phi'_k - (1-\xi)(\phi_i\phi'_k\psi''_j + \phi_i\phi''_k\psi'_j)]\Big\} d\xi,$$

$$F_{ijkl} = \int_0^1 \Big\{\frac{3}{2}(\varGamma - \varPi_0 - \varPi)\phi_i\phi'_j\phi'_k\phi'' - (8\phi_i\phi'_j\phi''_k\phi'''_l + 2\phi_i\phi'_j\phi'_k\phi^{(4)}_l + 2\phi_i\phi''_j\phi''_k\phi''_l)$$
$$-\gamma\left[\frac{1}{2}\phi_i\phi'_j\phi'_k\phi'_l - \frac{3}{2}(1-\xi)(\phi_i\phi'_j\phi'_k\phi''_l)\right]\Big\} d\xi.$$

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